# LQG Regulator and Kalman FIltering

F. Crevecoeur

August 8, 2013

**Exercise 1.** Consider the following control system:

$$\begin{aligned} \ddot{x} &= -k_v\dot{x} + F_x, \\ \ddot{y} &= -k_v\dot{y} + F_y, \\ \tau\dot{F}_x &= u_x + \lambda u_y - F_x, \\ \tau\dot{F}_y &= u_y + \lambda u_x - F_y, \end{aligned} \tag{1}$$

where $k_v$, $\tau$ and $\lambda$ are parameters. Express the control system in the form $x_{k+1} = Ax_k + Bu_k$. Use the property that any $n^{\text{th}}$ order linear ODE (Ordinary Differential Equation) can be transformed into a $n$-dimensional first order ODE as follows:

$$\begin{aligned} u^{(n)} &= a_0u + a_1u' + \ldots + a_{n-1}u^{n-1} \\ &\Leftrightarrow \\ \begin{bmatrix} u' \\ u'' \\ \vdots \\ u^{(n)} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix} \begin{bmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{bmatrix}. \end{aligned}$$

Use Euler integration scheme with $\delta t = 10$ ms. The state vector must be augmented with the stationary target vector (i.e. $\dot{\mathbf{x}}^* = 0$):

$$\begin{aligned} \mathbf{x}_k = \Big[ & x(k\delta) \quad y(k\delta) \quad \dot{x}(k\delta) \quad \dot{y}(k\delta) \quad F_x(k\delta) \quad F_y(k\delta) \\ & x^* \quad y^* \quad \dot{x}^* \quad \dot{y}^* \quad F_x^* \quad F_y^* \Big]^T. \end{aligned}$$

Adjust $A$ and $B$ according to the augmented state vector.

**Exercise 2.** Implement the backward recursion that give the optimal feedback gains:

$$
\begin{aligned}
L_k &= \left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A, \\
S_k &= Q_k + A^T S_{k+1}(A - BL_k), \\
s_k &= s_{k+1} + \mathrm{tr}(S_{k+1}\Omega_\xi), \\
S_N &= Q_N, \quad s_N = 0.
\end{aligned}
$$

Simulate the discrete dynamics of the control system with $R = \mathrm{diag}[10^{-5} \quad 10^{-5}]$, $Q_k = 0, \quad k = 1, 2, \ldots, N-1$ and $Q_N$ such that:

$$
\begin{aligned}
\mathbf{x}_N^T Q_N \mathbf{x}_N &= w_1(x_N - x^*)^2 + w_2(y_N - y^*)^2 \\
&\quad + w_3(\dot{x}_N - \dot{x}^*)^2 + w_4(\dot{y}_N - \dot{y}^*)^2.
\end{aligned}
$$

Use a noise covariance matrix $\Omega_\xi$ that affects the forces derivatives with variance $10^{-4}$. Simulate the system dynamics.

**Exercise 3.** Implement the forward recursion of a Kalman filter in the predictive case:

$$
\begin{aligned}
\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - H\hat{x}_k), \\
K_k &= A\Sigma_k H^T(H\Sigma_k H^T + \Omega_\omega)^{-1}, \\
\Sigma_{k+1} &= \Omega_\xi + (A - K_k H)\Sigma_k A^T.
\end{aligned}
$$

Apply it to the control system with the observation matrix $H = I_p$. Simulate the control and state estimation processes by applying the feedback gains to the estimated state (use $\Omega_\omega \sim 10^{-4} I_p$ and $\Sigma_1 = 10^{-6} I_n$).

**Exercise 4.** Vary the parameters $w_i$ to observe the minimum intervention principle. Add signal dependent noise in the optimal feedback gains (fully observable case) and observe the speed-accuracy trade-off by varying the time horizon ($N$).