Linear Systems Theory

in SensoriMotor Control

(see Syllabus JvO, sections 1.1-1.5)

The vestibular collic (neck) reflex of the owl in action

Psychophysics and electrophysiology both use the mathematical techniques from Systems Theory to understand the relation between structure and function from input-output relationships.

Systems theory as a

Black-Box Description:

‘Known’ **‘Unknown’** **‘Known’**

Linear Systems:

MUST obey the *Superposition Principle (SP):*

If input $x_1(t)$ => Output $y_1(t)$

and input $x_2(t)$ => Output $y_2(t)$

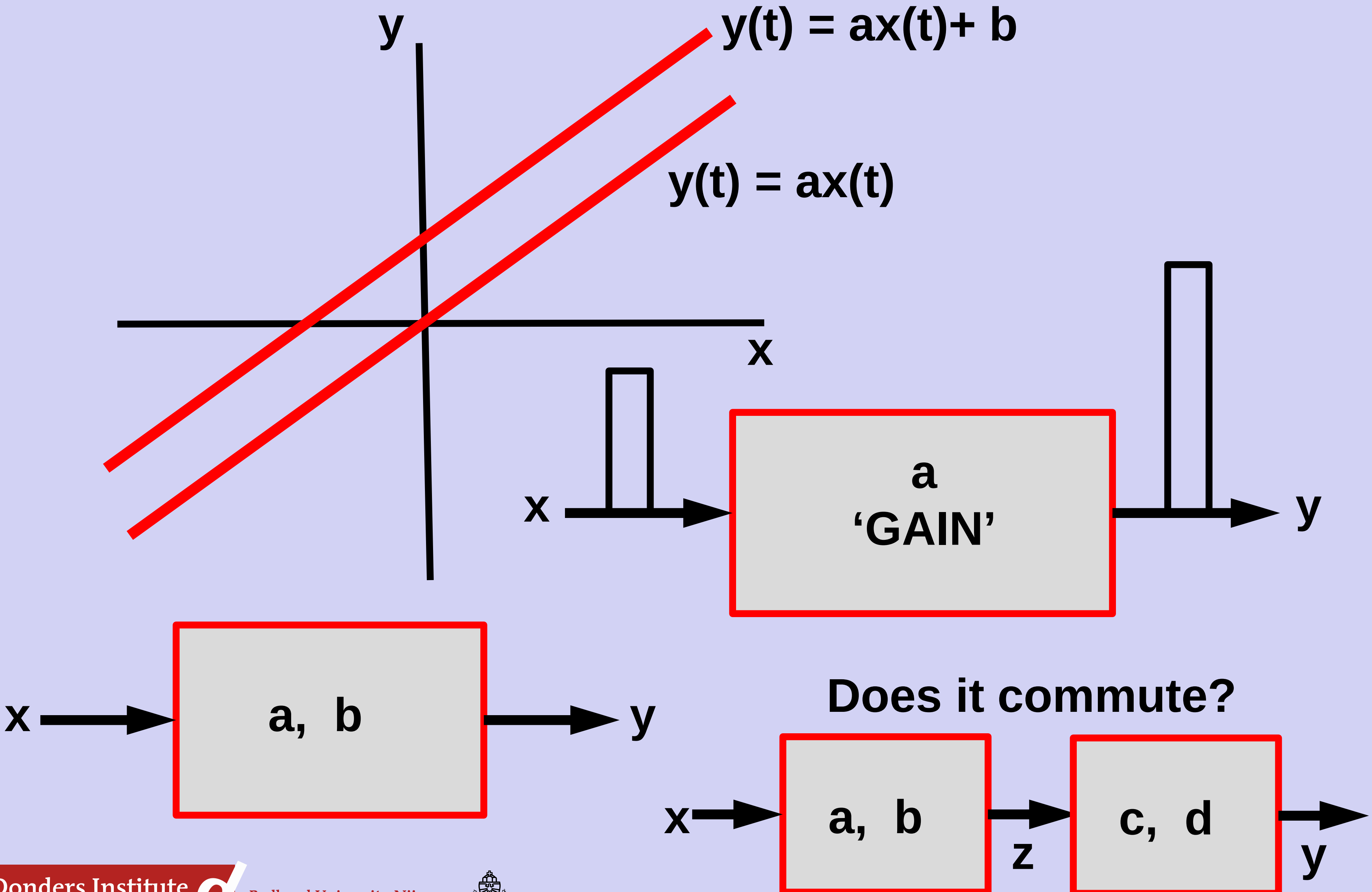
then it is required that:

$a\,x_1(t) + b\,x_2(t)$ => $a\,y_1(t) + b\,y_2(t)$

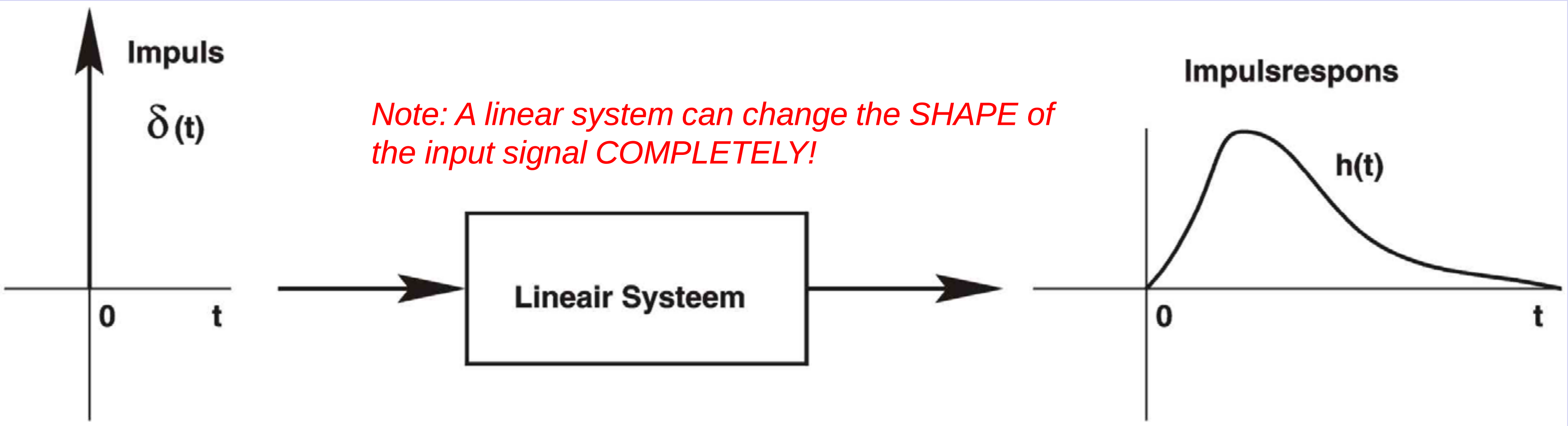
In general: $$\boxed{\sum_{n=1}^{N} a_n x_n(t) \Rightarrow \sum_{n=1}^{N} a_n y_n(t)}$$

Note: this requirement must hold for all t!

First shock: is this system linear? (Exercises 1+2)



Central concept of LS theory: the Impulse response



Crucial idea:

Because of the SP, the response of the LS to an arbitrary input can be computed from the system's impulse response!

How?

An exact description of signal x(t) with Dirac-impulse functions is as follows:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \delta(t-\tau) \cdot d\tau$$

Think of the Dirac pulse as the equivalent of a stroboscopic light pulse in the disco to observe/measure the dancing people….

This yields an *exact* prediction of response y(t) with the impulse response as building block! :

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

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- we assume a *causal* system (Exercise 3)
- this means that there is NO RESPONSE when there is NO INPUT

$$h(s) = 0 \quad for \quad s \leq 0$$

$$h(t-\tau) = 0 \quad for \quad \tau \geq t$$

$$y(t) = \int_{-\infty}^{t} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

From this it follows that:

$$\boxed{y(t) = \int_{0}^{+\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau}$$

the *CONVOLUTION INTEGRAL* *(see also Exercise 4!)*

$$y(t) = \int_{0}^{+\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

'now'

'current system's output'

'now'

'the start of the universe'

'the system's past input'

'the past'

'the system's dynamic memory of the past input'

'superposition over the past'

Example 1: (= Exercise 6)

Question: what is the step response of a linear system when its impulse response is h(t) ?

$$\boxed{y(t)=\int_{0}^{+\infty} x(t-\tau)\cdot h(\tau)\cdot d\tau}$$

x(t) = the unitary step: x(t)=1 for t≧0, 0 elsewhere

x(t) = 1 for t-τ ≧ 0, 0 elsewhere, so: τ ≤ t

Example 2: Suppose as input the harmonic function:

$$x(t) = \sin(\omega \cdot t)$$

Assume that the impulse response is h(t).

Then what is the system's output?

(= Exercise 8)

$$y(t) = \int_{0}^{+\infty} \sin(\omega \cdot (t - \tau)) \cdot h(\tau) \cdot d\tau$$

$$\sin(\omega t - \omega\tau) = \sin(\omega t) \cdot \cos(\omega\tau) - \cos(\omega t) \cdot \sin(\omega\tau)$$

$$y(t) = \sin(\omega t) \cdot \underbrace{\int_{0}^{+\infty} \cos(\omega\tau) \cdot h(\tau) \cdot d\tau}_{A(\omega)} - \cos(\omega t) \cdot \underbrace{\int_{0}^{+\infty} \sin(\omega\tau) \cdot h(\tau) \cdot d\tau}_{B(\omega)}$$

So, when the input to a LS is:

$$x(t) = \sin(\omega \cdot t)$$

Its output is:

$$\boxed{y(t) = G(\omega) \cdot \sin(\omega \cdot t + \varphi(\omega))}$$

- *This is again a harmonic function.*
- *Output amplitude and phase depend on the frequency of the input.*
- *The output frequency has not changed!* *(see Exercise 7)*
- *Harmonic functions are eigenfunctions of (all) LS*

This leads to an alternative characterization of a LS:

- **the AMPLITUDE CHARACTERISTIC G(ω) and**
- **the PHASE CHARACTERISTIC Φ(ω)**

Together: the TRANSFER CHARACTERISTIC

$$\boxed{H(\omega)}$$

Fourier analysis: H(ω) is the FT of h(τ)!

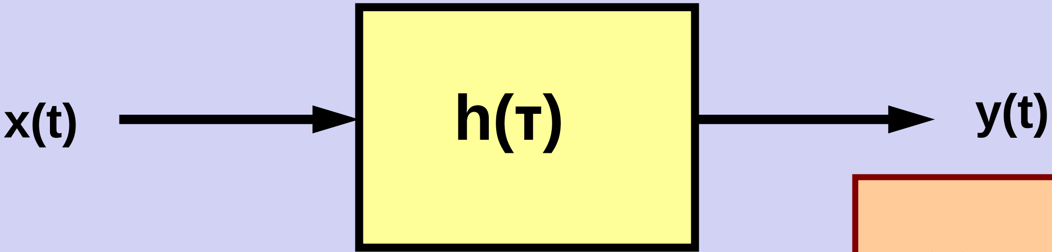
Fourier analysis of LS is mathematically very convenient:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot \exp(-i \cdot \omega t) \cdot dt$$

It then follows from the convolution integral: (see Exercise 9):

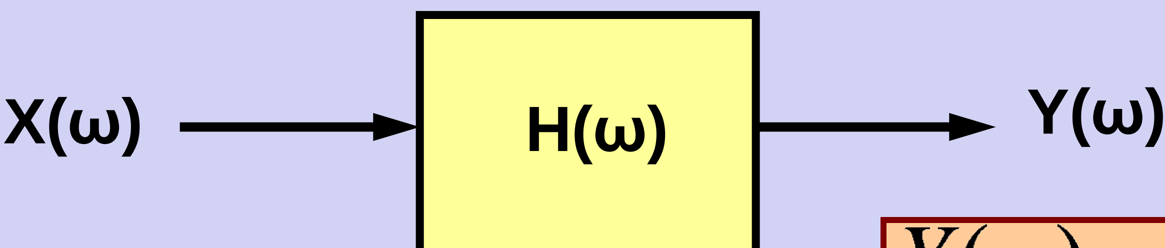
$$Y(\omega) = H(\omega) \cdot X(\omega)$$

Brief recapitulation: FOR ALL LINEAR SYSTEMS:



$$y(t) = \int_0^\infty h(\tau) \cdot x(t-\tau) \cdot d\tau$$

Measuring h(τ):
- directly from the impulse (often not possible ……)
- from the step response (and then take the derivative)
- with GWN (take the cross-correlation)
- or: after inverse Fourier transformation of
  the sine/cosine responses



$$Y(\omega) = H(\omega) \cdot X(\omega)$$