Answers to Exercises Syllabus Van Opstal Part 1

Systems Theory

Exercise 1:
Apply the Superposition demand:

$$y_{1+2}(t) = b + a \cdot (x_1(t) + x_2(t)) \neq y_1(t) + y_2(t) = 2b + a \cdot (x_1(t) + x_2(t))$$

So, only for $b = 0$ (line without an offset) will such a system be linear (it's then called a *'gain'*).

A second crucial property of linear systems is *commutativity*: if system A is followed by system B, the overall transformation should be the same when system A is followed by system B: or symbolically, $A \cdot B = B \cdot A$. We here show that for the general straight line, commutativity does *not* hold either:
Suppose that system A is given by: $y = ax + b$ and system B by: $y = cx + d$, with a, b, c, d all unequal parameters unequal to zero. First take what results from AB, then from BA, and verify whether or not they are equal:

- output of A: $z = ax + b$ is the input to B, so $y = c \cdot z + d = c \cdot (ax + b) + d = ac \cdot x + bc + d$
- output of B: $z = cx + d$ is the input to A, so $y = a \cdot z + b = a \cdot (cx + d) + b = ac \cdot x + ad + b$

and these results are unequal! They will only be equal when the two offsets, $b = d = 0$!

Exercise 2:
Apply the Superposition demand:

$$y_{1+2}(t) = a \cdot (x_1(t) + x_2(t))^2 + b = ax_1(t)^2 + ax_2(t)^2 + 2a \cdot x_1(t) \cdot x_2(t) + b \neq$$

$$y_1(t) + y_2(t) = 2b + a \cdot (x_1(t)^2 + x_2(t)^2)$$

Now nothing can save this system! Linearity is killed by both the offset ánd the cross-term....

Exercise 3:
From $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$ we demand causality: $h(x) = 0$ for $x < 0$, so $h(t - \tau) = 0$ for $t - \tau < 0$, which is for $\tau > t$. The integral thus becomes:

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t - \tau)d\tau$$

We then introduce a new variable to integrate (to avoid confusion of symbols), by taking: $t - \tau \equiv \sigma$, so that $d\tau = -d\sigma$ and for $\tau = t \Rightarrow \sigma = 0$ and $\tau = -\infty \Rightarrow \sigma = +\infty$, so that $y(t) = \int_{\infty}^{0} h(\sigma)x(t - \sigma)(-d\sigma)$ or:

$$y(t) = \int_{0}^{\infty} h(\tau)x(t - \tau)d\tau$$

Exercise 4:
Really try to understand all symbols of the convolution equation!:
t is the time point at which we observe the output. Let's call this the 'present'. The 'shift-time'

τ that figures in the integral is therefore a look into the 'past'. Indeed, τ is positive (so the value of zero = 'now', and infinity is the 'start of the universe'.). The impulse response function $h(\tau)$ is therefore a measure for how strong the past input (at $t-\tau$) has to be weighted to explain the output at time 'now' (it's the system's 'memory'!). Because $h(\tau)$ is a time-dependent function, this memory is in general 'dynamic'.

Exercise 5:
For the gain, the following must hold:

$$y(t) = \int_0^\infty x(t-\tau)\cdot h(\tau)d\tau \equiv a\cdot x(t)$$

(the left-hand side holds for all linear systems, the right-hand side is our particular linear system). This equation can only be true (*for all times, t*) when:

$$h(\tau) = a\cdot\delta(\tau)$$

Indeed, in that case:

$$y(t) = \int_0^\infty x(t-\tau)\cdot a\cdot\delta(\tau)d\tau = a\cdot x(t)$$

because the only value for τ that doesn't let the impulse response vanish on the left-hand side is $\tau = 0$!

Exercise 6:

$$y(t) = \int_0^\infty x(t-\tau)\cdot h(\tau)d\tau = \int_0^\infty U(t-\tau)\cdot h(\tau)d\tau = \int_0^t 1\cdot h(\tau)d\tau$$

done! (note that $U(t-\tau)$ is 1 as long as its argument $(t-\tau) > 0$, so $\tau \leq t$).

Exercise 7:
Substitute for the input: $x(t) = \sin(\omega_0 t)$.

(a) $y(t) = a\sin(\omega_0 t) + b$. The spectrum of the output contains the following *two* frequencies: $\omega \in \{0, \omega_0\}$. In other words, the DC is added by the system, and this is forbidden for linear systems!

(b) $y(t) = a\sin(\omega_0 t)^2 + b$.We now note that this can be rewritten as: $y(t) = \frac{a}{2}\cdot(\cos(2\omega_0 t) - 1) + b$, so that the spectrum of the output is: $\omega \in \{0, 2\omega_0\}$. None of these output frequencies were present in the input, so the system cannot be linear!

Exercise 8:
An important consequence of the superposition principle is that harmonic functions are *Eigenfunctions* of the linear system. Therefore, the shape of the function remains unaltered by the system's transformation! (sine in gives sine out!). This is an important test for the linearity of systems: if a pure sinusoid as input yields an output with harmonic distortions (e.g. other frequencies are added to the output), then there must be a nonlinearity in the system. Here we prove this very important eigenfunction principle (which at the same time is a derivation for the Fourier integral ...). We take:

$$y(t) = \int_0^\infty h(\tau)\cdot\sin(\omega(t-\tau))d\tau$$

and elaborate: $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$, dus

$$y(t) = \sin(\omega t) \cdot \int_0^\infty h(\tau) \cdot \cos(\omega\tau) d\tau - \cos(\omega t) \cdot \int_0^\infty h(\tau) \cdot \sin(\omega\tau) d\tau$$

since both integrals can be reduced to functions that only depend on ω, we write:

$$y(t) = A_1(\omega)\sin(\omega t) + A_2(\omega)\cos(\omega t)$$

Suppose $A_1(\omega) \equiv A(\omega) \cdot \cos(\Phi(\omega))$ and $A_2(\omega) = A(\omega)\sin(\Phi(\omega))$, with $\Phi(\omega) = \arctan(A_2(\omega)/A_1(\omega))$ en $A(\omega) = \sqrt{A_1^2(\omega) + A_2^2(\omega)}$, we then finally obtain:

$$y(t) = A(\omega)\sin[\omega \cdot t + \Phi(\omega)]$$

and this is indeed a sinusoid with the $SAME$ frequency as the input, but with an amplitude and phase that depend on the frequency and the impulse response of the system. In fact, these turn out to be the amplitude and phase of the Fourier transform of the impulse response function!

Exercise 9:
Substitution of the convolution integral in the Fourier Transform of $y(t)$ yields:

$$Y(\omega) = \int_{-\infty}^{\infty} [\int_0^\infty h(\tau)x(t-\tau)d\tau] \exp(-i\omega t)dt = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau] \exp(-i\omega t)dt$$

(note that $h(\tau) = 0$ for $\tau < 0$!)

$$= [\int_{-\infty}^{\infty} h(\tau)\exp(-i\omega\tau)d\tau] \cdot [\int_{-\infty}^{\infty} x(t-\tau)\exp(-i\omega(t-\tau))d(t-\tau)] = H(\omega) \cdot X(\omega)$$

ready! This is an extremely important result, so it's worth your time to try to really understand it!

Exercise 10: The first-order Low-Pass Filter ('Leaky Integrator').
Input $x(t)$, output is the voltage across the condensator, $y(t)$. According Kirchhoff's law of voltages:

$$x(t) = V_R + V_C = R \cdot i(t) + \frac{q(t)}{C} = R \cdot \frac{dq}{dt} + \frac{q}{C}$$

The impuls response is found by taking $x(t) = \delta(t)$, but this is not easy to solve because of the infinities. We therefore take the step response. At $t = 0$ we apply voltage V_0. The diff. eqn. then becomes:

$$R \cdot \frac{dq}{dt} + \frac{q}{C} = V_0$$

(if $t \geq 0$). Solution of the homogeneous diff. eqn. is

$$\frac{dq}{dt} = -\frac{1}{RC}q(t) \Rightarrow q_0(t) = k \cdot \exp(-\frac{t}{RC})$$

A particular solution is a constant: $q_p = V_0 \cdot C$. Total solution is the sum:

$$q(t) = V_0 \cdot C + k \cdot \exp(-\frac{t}{RC}) = V_0 \cdot C(1 - \exp(-\frac{t}{RC}))$$

(use boundary condition $q(0) = 0$). The voltage across the condensator (the step response) is therefore ($V = q/C$):

$$V_C(t) \equiv s(t) = V_0 \cdot (1 - \exp(-\frac{t}{RC}))$$

The impulse response is the derivative of the step response:

$$h(t) = \frac{1}{V_0}\frac{ds}{dt} = \frac{1}{RC}\exp(-\frac{t}{RC})$$

The transfer characteristic of the LP system is found by Fourier Transformation of the impulse response (note that $t > 0$ because of the causality constraint):

$$H(\omega) = \int_0^\infty h(t) \cdot \exp(-i\omega t)dt = \frac{1}{RC}\int_0^\infty \exp(-\frac{t}{RC} - i\omega t)dt$$

Thus:

$$H(\omega) = \frac{1}{RC} \cdot \int_0^\infty \exp(-t\frac{1+i\omega RC}{RC})dt = \frac{1}{RC} \cdot \frac{RC}{1+i\omega RC}\exp(-t\frac{1+i\omega RC}{RC})\Big|_0^\infty = \frac{1}{1+i\omega RC}$$

See also the syllabus for the amplitude and phase characteristics: first, take the real and the imaginary parts:

$$\frac{1}{1+i\omega RC} = \frac{1}{1+i\omega RC} \cdot \frac{1-i\omega RC}{1-i\omega RC}$$
$$= \frac{1}{1+\omega^2R^2C^2} - i \cdot \frac{i\omega RC}{1+\omega^2R^2C^2}$$

etc.

$G(\omega) = \sqrt{\mathsf{Re}(H)^2 + \mathsf{Im}(H)^2}$ and $\Phi(\omega) = \arctan\left(\frac{\mathsf{Im}H}{\mathsf{Re}H}\right)$

Exercise 11: First-Order High-Pass system

The simplest way to determine the impulse response of the High-Pass system is by using the idea (from Kirchhoff!) that *High-Pass = All-Pass - Low-Pass* (but, of course, we can also use the model in Fig. 6, see below). Note that an all-pass system would leave the Dirac impulse unchanged! Thus:

$$h(\tau) = \delta(\tau) - \frac{1}{RC} \cdot \exp\left(-\frac{\tau}{RC}\right) \tag{1}$$

(the first term, $\delta(\tau)$, gives the unchanged Dirac input pulse, 'all pass').
The step response is the integral of the impulse response, so:

$$s(\tau) = \int_0^\tau h(t)dt = \exp\left(-\frac{\tau}{RC}\right) \tag{2}$$

In contrast to the LP-system the HP system does NOT transmit the d.c. component: ($\lim_{\tau\to\infty} s(\tau) = 0$), whereas the instantaneous step at $t = 0$ is left unharmed!

Alternative derivation: use the electric circuit, just as in the LP model, as shown in Fig. 6:
We determine the step response to ($x(t) = V_0$) for the HP system from Kirchhoff's law:

$$x(t) = \frac{q}{C} + R \cdot \frac{dq}{dt} \quad y(t) = R\frac{dq}{dt}$$

Because for $q(t)$ we have the same solution as for the LP model (it's the same electrical circuit!) we now find for the output across the resistor that:

$$s(t) = R\frac{dq}{dt} = V_0 \exp(-t/RC)$$

The impulse response then is the derivative of this step response:

$$h(t) = \delta(t) - \frac{1}{RC}\exp(-t/RC)$$

Why the $\delta(t)$? Remember that at $t = 0$ the input jumps immediately to V_0, and that for $t < 0$ it is zero! Thus, a better mathematical description for the step response is:

$$s(t) = U(t) \cdot V_0 \exp(-t/RC)$$

with $U(t)$ the step function. The derivative:

$$h(t) = \frac{dU}{dt}V_0 \exp(-t/RC) - U(t)\frac{V_0}{RC}\exp(-t/RC)$$

and remember that dU/dt=0 except in $t = 0$, where it equals $\delta(t)$, and that $U(t) = 1$ for $t > 0$ so we're done!
Fourier analysis gives the same function as above (with a minus sign) plus the FT of the delta-function, which equals 1:

$$H(\omega) = 1 - \frac{1}{1 + i\omega RC} = \text{ALL} - \text{LowPass} = \frac{i\omega RC}{1 + i\omega RC}$$

You can now verify that the amplitude and phase characteristics are:

$$G(\omega) = \frac{\omega T_c}{\sqrt{1 + \omega^2 T_c^2}} \quad \text{en} \quad \Theta(\omega) = \arctan\left(\frac{1}{\omega T_c}\right) \tag{3}$$

As expected, $G(\omega) \to 1$ for $\omega \to \infty$, and to 0 for $\omega \downarrow 0$. The output's phase now always advances ('phase-lead') between $\pi/2$ (cosine becomes sine) and 0 (exact synchronous).

For the next three (important, because they occur very frequently in models!) examples the Transfer Characteristics can be calculated without having to do explicit Fourier Analysis! Remember what the Transfer Characteristic really is: it's the response of the system to a pure sinusoid with an arbitrary frequency ω!

Exercise 12: The ideal Integrator.
The response of the integrator to a sinusoidal input is given as $y(t) = \int_0^t \sin(\omega\tau)d\tau$. Since integration is a limit case of summation, it will not come as a surprise that this is a linear process. A system performing time integration is therefore a linear system, and therefore obeys convolution:

$$y(t) = \int_0^t h(\tau)x(t-\tau)d\tau \equiv \int_0^t x(\tau)d\tau \quad \Rightarrow \quad h(\tau) = 1 \;\; \forall \tau \geq 0 \;\; \text{and} \;\; s(\tau) = \tau \;\; \forall\, \tau \geq 0 \tag{4}$$

Now verify that the frequency characteristics for the integrator are given by:

$$G(\omega) = \frac{1}{\omega} \text{ and } \Theta(\omega) = -\pi/2 \ \ \forall\, \omega \tag{5}$$

In summary: a pure integrator introduces a phase lag of -90^o, and its amplitude characteristic declines inversely proportional with frequency (Fig. 36, fourth row). Note also that this characteristic (at least for high frequencies) describes a LP filter.

A LP filter is also called a *leaky integrator* in the literature (note that the pure integrator has a time constant of $T_c = \infty$, i.e. an infinitely long memory. In a LP filter information from the past 'leaks away' with a time constant T_c sec. Only a constant signal (a d.c.) is integrated 'purely' by a LP fitler.

Exerxise 13: The ideal Differentiator.
The response of the differentiator is $y(t) = d\sin(\omega t)/dt$!
Also differentiation is a linear process (it follows superposition principle), and therefore a differentiator is a linear system. The derivative of the Dirac pulse, however, is a little strange: it's a Dirac pulse, immediately followed (at $t = 0$) by a negative Dirac pulse! This is a so-called *'doublet'* and can be made qualtitatively understandable by approximating the Dirac pulse by a high rectangular pulse. In the Figure we omitted this function, but show the step- and ramp-responses instead. Verify that the defining characteristics of the differentiator are described by:

$$s(\tau) = \delta(\tau) \quad G(\omega) = \omega \text{ en } \Theta(\omega) = \pi/2 \tag{6}$$

The amplitude characteristic increases linearly with frequency, and the phase lead is always $+90^o$. The ramp-response of the differentiator is the Heaviside function. In this case, the high-frequency behavior of the HP filter resembles that of the pure differentiator.

Exercise 14: The pure time delay.
The response of the delay to a sine wave is by definition: $y(t) = \sin(\omega(t - \Delta T))$. We hence get:

For the pure delay *all* signals are passed without any deformation (it's an 'all pass' system!), only with a lag of ΔT s. The impulse response and step response of the delay are therefore given by:

$$h(\tau) = \delta(\tau - \Delta T) \text{ and } s(\tau) = H(\tau - \Delta T) \quad \text{for } \tau \geq \Delta T \tag{7}$$

The amplitude characteristic is always 1: $G(\omega) = 1 \ \ \forall\, \omega$, but now verify that the phase characteristic has the following interesting behaviour:

$$\Theta(\omega) = -\omega \Delta T \tag{8}$$

In other words, the phase-lag increases linearly with freqyency. Is this in line with your intuition?

Exercises 15+16:
To create a **bandpass filter** (BP) we place a LP and a HP filter in series, but we should take care that the respective frequency bands overlap (and therefore the cut-off frequencies are positioned correctly). The time constants thus:

$$T_{\text{low}} < T_{\text{high}} \Rightarrow (R_L C_L) < (R_H C_H)$$

The difference in cut-off frequencies then determines the bandwidth of the BP filter:

$$BW = \frac{1}{R_L C_L} - \frac{1}{R_H C_H}$$

The transfer characteristic of the filter then is:

$$H(s) = \frac{sT_H}{(1+sT_L)(1+sT_H)}$$

with two poles (see later!), in $s = -1/T_L$ and $s = -1/T_H$, respectively, with a d.c.-gain and an $\omega = \infty$ transfer of 0 (as it should be). For the amplitude characteristic:

$$\|H(s)\| = \|H(s)_L\| \cdot \|H(s)_H\| = \frac{\omega T_H}{\sqrt{1+\omega^2 T_L^2}\sqrt{1+\omega^2 T_H^2}}$$

for which you can verify that this is maximal at:

$$\omega_o = \frac{1}{\sqrt{T_L \cdot T_H}}$$

For a **band-stop** (BS) filter we demand the opposite: now the cut-off point of the LP filter should fall outside the HP cut-off (no overlap), but for any transfer at all, the two filters need to be placed parallel to each other! Now the transfer characteristic is:

$$H(s) = \frac{1}{(1+sT_L)} + \frac{1}{(1+sT_H)}$$

for which the amplitude and phase behavior can be readily assessed by applying some calculus (e.g., in Mathematica of MatLab!).
An **All-Pass** system can be made by putting the HP and LP filters in parallel, but with their cut-off frequencies such the same as for the BP filter: so that the LP and HP bands will overlap: time constant of the LP filter is therefore shorter than for the HP filter. For the transfer characteristic we obtain the same relation as for the BS filter, but the time constants were chosen differently, and therefore the filter has different properties.

Exercise 17:
The eye that is patched (but otherwise unharmed) gets the constant retinal-slip speed of the stimulus as an input, because the paralyzed, but viewing, eye cannot move. As a result, the patched eye will respond with a movement as if there was no feedback, because its movement does not alter the visual input! Indeed, in this way one can measure the open-loop gain of the system.

Exercise 18:
We need to sum two different input signals that together yield a constant output: The first input is a constant head velocity, $\dot{H}_0$, which yields a decaying exponential as output (it's the HP response of the VOR to a step input):

$$\dot{E}_{S1}(t) = -\dot{H}_0 \cdot \exp(-t/T_{VOR})$$

The second input is a *constant acceleration*. Call the acceleration $\ddot{H}(t) = \alpha$ deg/s^2, which is the integral of a constant velocity. Note that if we perform a linear operation on the input (e.g. integration), then we can perform the same operation on the output (because of the superposition principle!). Therefore, integration of the constant acceleration vestibular input with strength α

yields input head velocity $\dot{H}(t) = \alpha \cdot t$ (a ramp stimulus). The resulting output is then given by integration of the previous result:

$$\dot{E}_{S2}(t) = -\alpha T_{\text{VOR}}(1 - e^{-t/T_{\text{VOR}}})$$

By adding the two inputs, the two outputs will also add (superposition!), and it will be possible to get a constant output by tuning α. So, the total input is:

$$\dot{H}(t) = \dot{H}_0 + \alpha \cdot t$$

to which the VOR will respond with:

$$\dot{E}_S(t) = -\dot{H}_o \cdot e^{-t/T_{\text{VOR}}} - \alpha T_{\text{VOR}}(1 - e^{-t/T_{\text{VOR}}}) = -(\dot{H}_o - \alpha T_{\text{VOR}})e^{-t/T_{\text{VOR}}} - \alpha T_{\text{VOR}}$$

The time-dependent term will vanish provided $(\dot{H}_o - \alpha T_{VOR}) = 0$, so:

$$\alpha = \frac{\dot{H}_o}{T_{\text{VOR}}}$$

Exercise 19:
(a) Suppose that the LT of $x(t)$ is given by $X(s)$, then the delayed signal becomes:

$$Y(s) = \int_0^\infty x(t - \Delta T)e^{-s\cdot t}dt = \int_0^\infty x(t - \Delta T)e^{-s\cdot(t-\Delta T)} \cdot e^{-s\Delta T}dt = X(s) \cdot e^{-s\Delta T}$$

The transfer function for a delay is therefore $H \equiv Y/X$:

$$H(s) = \exp(-s\Delta T)$$

In the frequency domain we have: substitute $s = j\omega$:

$$H(\omega) = \exp(-j\omega\Delta T)$$

which has an amplitude characteristic $|H(\omega)| = 1 \;\; \forall\omega$ and phase characteristic $\Phi(\omega) = -\omega\Delta T$.

(b) For the transfer function of the total feedback system we obtain:

$$H(s) = \frac{A\exp(-s\Delta T)}{1 + sT + A\exp(-s\Delta T)}$$

The loop gain is given by the product of all systems in the loop:

$$L(s) = \frac{A\exp(-s\Delta T)}{1 + sT}$$

for which the amplitude characteristic is:

$$|L(\omega)| = \frac{A}{\sqrt{\omega^2 + T^2}}$$

and the phase characteristic:

$$\Phi(\omega) = -\omega\Delta T - \arctan(\omega T)$$

Unstable behaviour of the system occurs when $\Phi(\omega_0) = -180^o$ and $|L(\omega_0)| > 1$. It is convenient to try to approximate this numerically. T

(c) Lowering A or increasing T brings the gain below 1 and the instability disappears. This may be important for applications, because in this way the system can be kept from instability for its normal working range. A disadvantage is that the bandwidth of the system then decreases (the BW is determined at the point where log(Gain) crosses zero). Increasing A, or lowering T shifts the instability point to higher frequencies. If these frequencies fall beyond the operation range, this could also be an attractive solution to prevent instability.

Note that if $\Delta T \ll T$, the phase curve of the delay moves rightward, and the gain of the system remains below 1 (0 dB), preventing instability. However, when the delay approaches the time constant T the situation becomes problematic! In the CNS such situations could occur. In that case, other means are required to keep the system stable (e.g. parametric feedback, which means that the system parameters dynamically change according to the conditions, to prevent instability. This topic falls beyond the scope of this lecture).

Model of the Oculomotor plant:

The oculomotor plant (eye muscles, globe and surrounding fat tissues) is often modeled by a second-order low-pass filter (see afternoon session, and second part of the Syllabus!), which is obtained by a series concatenation of two first-order low-pass filters with different time constants. Just for fun, and only once, we will calculate explicitly through convolution, how to obtain the combined impulse response of this simple serial system (and appreciate immediately, why from now on we will use Fourier and Laplace descriptions instead!).

(a) For a series concatenation of two first-order LP filters with different time constants we obtain in the time domain the following convolution relations:

$$y(t) = \int_0^\infty h_2(\tau)u(t-\tau)d\tau$$

with in turn:

$$u(t) = \int_0^\infty h_1(\sigma)x(t-\sigma)d\sigma$$

For $x(t)$ we choose the Dirac impulse function, so that $u(t) = h_1(t)$.
Substitute, so that the impulse response of the total system, $h(t)$, is:

$$h(t) = \int_0^\infty h_2(\tau)h_1(t-\tau)d\tau$$

and with $h_1(t-\tau) = 0$ for $\tau > t$ (causality) the integration boundaries become $[0, t]$.
Now substitute the expressions for the two first-order impulse-response functions:

$$h_1(t) = F_1 \exp(-F_1 \cdot t) \quad \text{en} \quad h_2(t) = F_2 \exp(-F_2 \cdot t)$$

with $F_i \equiv 1/T_i$. It then follows for the total impulse response:

$$h(t) = \frac{1}{T_1 - T_2} \cdot \left(\exp(-\frac{t}{T_1}) - \exp(-\frac{t}{T_2}) \right)$$

This function is 0 for $t = 0$ and approaches 0 for $t \to \infty$. For specific t (which?) th impulse response reaches its only maximum. This is all left as exercise to the reader.
The conceptual meaning of $h(\tau)$ to the CNS is as follows: a nerve impulse (a 'spike', with a duration in the order of 1 ms) is for a slow system like the eye-eyemuscle plant (with a response time in the order of many tens of ms) in good approximation a Dirac-impulse function. The impulse response of the system is therefore a good approximation of the behavior of the eye in response to a single action potential. According to this the eye will reach a maximum amplitude after a brief while (after a couple of time constants T_2), after which the eye will slowly drift back with a long time constant (order T_1) to the equilibrium position. This backward drift is the reason that the eye-eyemuscle system requires a 'step' input signal in order to keep the eye in a steady peripheral fixation position (see syllabus for further discussion of this topic).
The step response of this system is given by integration of the impulse response:

$$s(\tau) = \int_0^\tau h(\sigma)d\sigma = \frac{T_1}{T_1 - T_2}[1 - \exp(-\tau/T_1)] - \frac{T_2}{T_1 - T_2}[1 - \exp(-\tau/T_2)]$$

a function which in its increasing behavior shows the influence of the two time constants.

(b) The transfer characteristic is described by (for the time-being we use: $s = i\omega$, until we have dealt with the Laplace transform):

$$H(s) = \frac{1}{(1 + sT_1)(1 + sT_2)}$$

It's behavior is LP (approaches zero for $\omega = \infty$, and 1 for d.c.), and is characterized by two cut-off points because of the two time constants. Using the remarks in the syllabus it's straightforward to construct the resulting Bode plot for this system from the two individual first-order filters ('summing of characteristics').