Laplace Transformation,

Pole-Zero analysis,

the Oculomotor Plant

and the VOR, revisited

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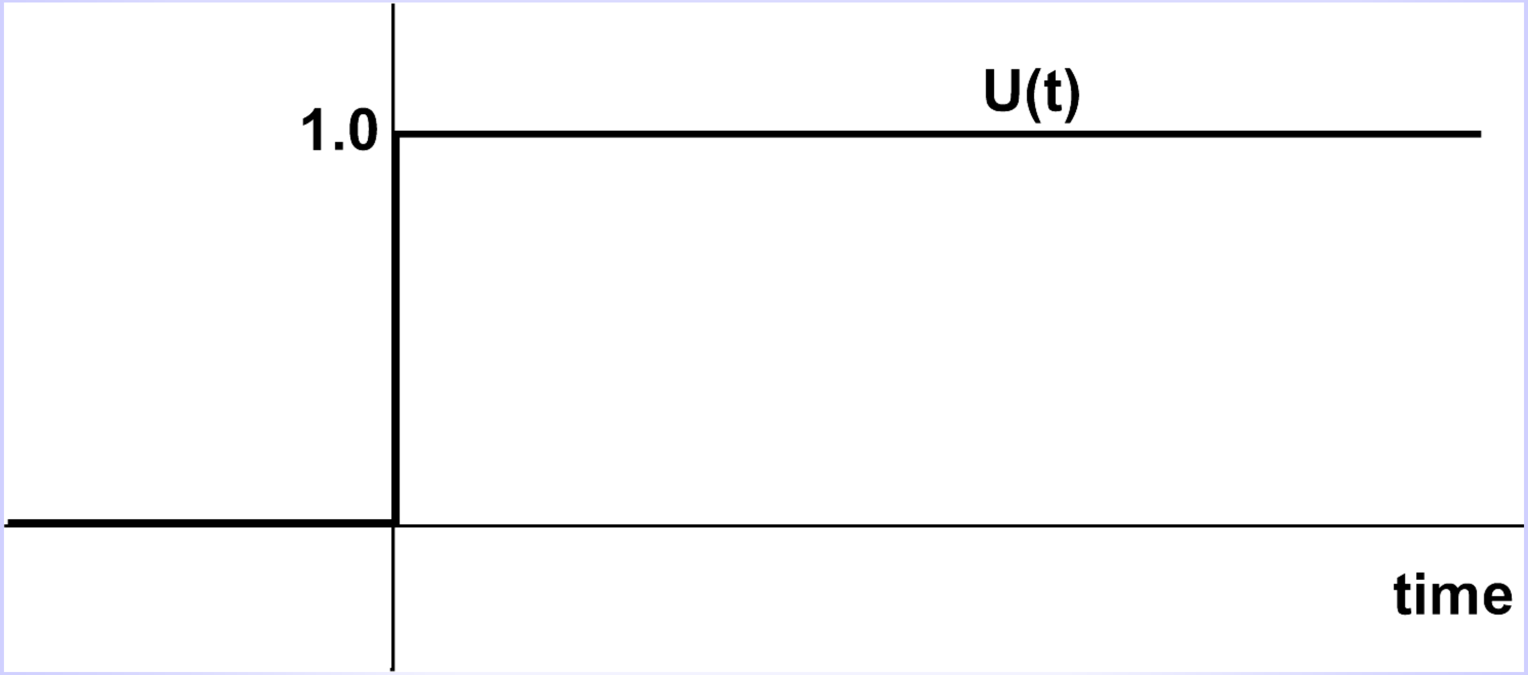
Problem: the Fourier transform of many, often used functions is not defined! E.g., the **step function**:



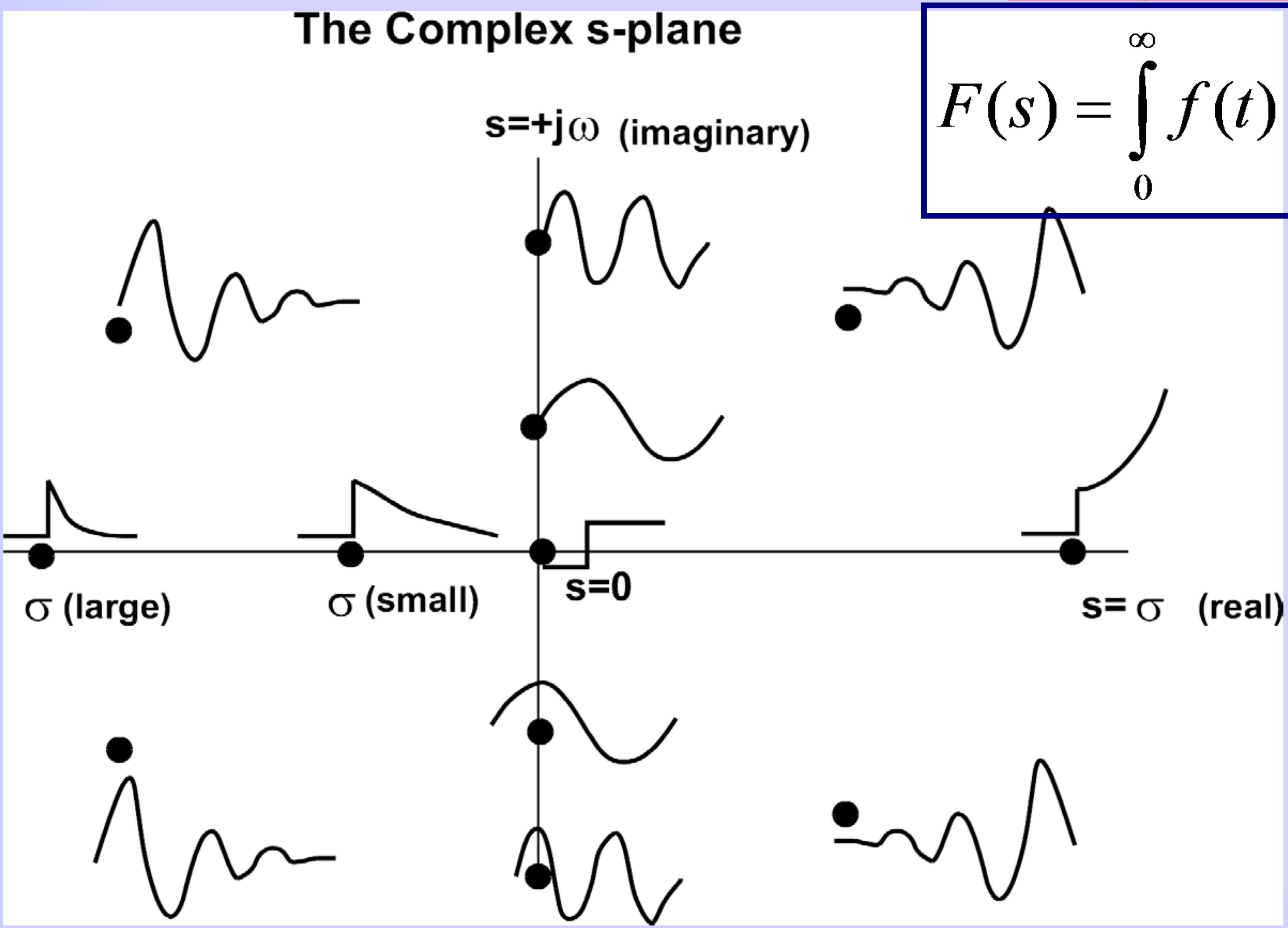
$$F(\omega)=\int_{-\infty}^{\infty}U(t)e^{-j\omega t}dt=\int_{0}^{\infty}e^{-j\omega t}dt=-\frac{1}{j\omega}e^{-j\omega t}\Big|_{0}^{\infty}=??$$

'weighting function'

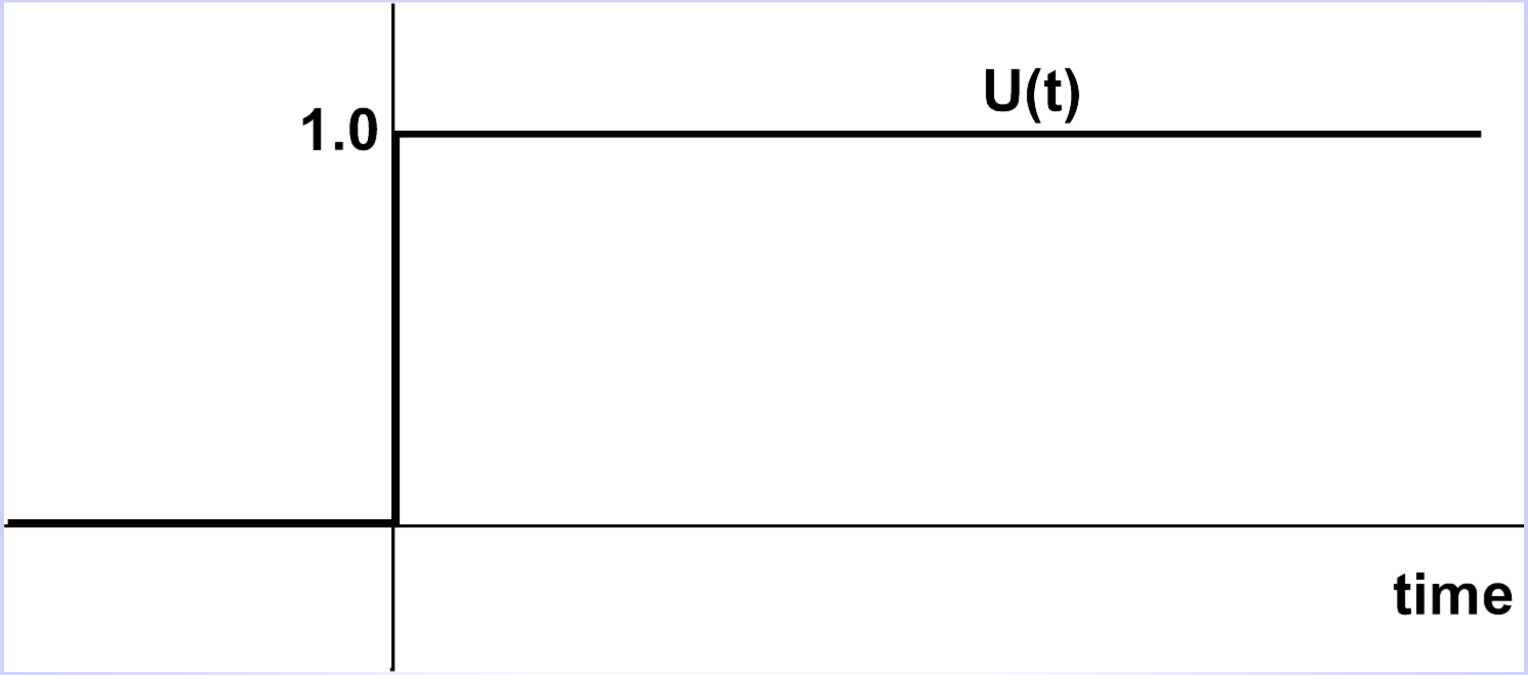
Laplace transform: uses a new weighting function!

$$e^{-st} \qquad s \equiv \sigma + j\omega$$

$$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$



The Laplace transform for the step function:



$$F(s) = \int_0^\infty U(t) \cdot e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

If $f(t)$ en $g(t)$ -> $F(s)$ en $G(s)$

then (left as Exercises for you ….):

	$a \cdot f(t)$	->	$a \cdot F(s)$	scaling
☛	$f(t \pm a)$	->	$\exp(\pm a \cdot s) \cdot F(s)$	delay
☛	df/dt	->	$s \cdot F(s)$	derivative
☛	$\int f(t) \cdot dt$	->	$F(s)/s$	integral
☛	$\exp(\pm a \cdot t)$	->	$1/(s \mp a)$	exponential
	$U(t)$	->	$1/s$	step
	$\delta(t)$	->	1	Dirac pulse
☛	$\exp(\pm a \cdot t) \cdot f(t)$	->	$F(s \mp a)$	exp. decay
	$h(t)=\int f(x) \cdot g(t-x) \cdot dx$	->	$H(s) = F(s) \cdot G(s)$	convolution

We apply the Laplace transform to the (1st-order) model of the eye muscles (the 'oculomotor plant'):

$$\Delta R_m(t) = k \cdot E(t) + r \cdot \frac{dE}{dt}$$

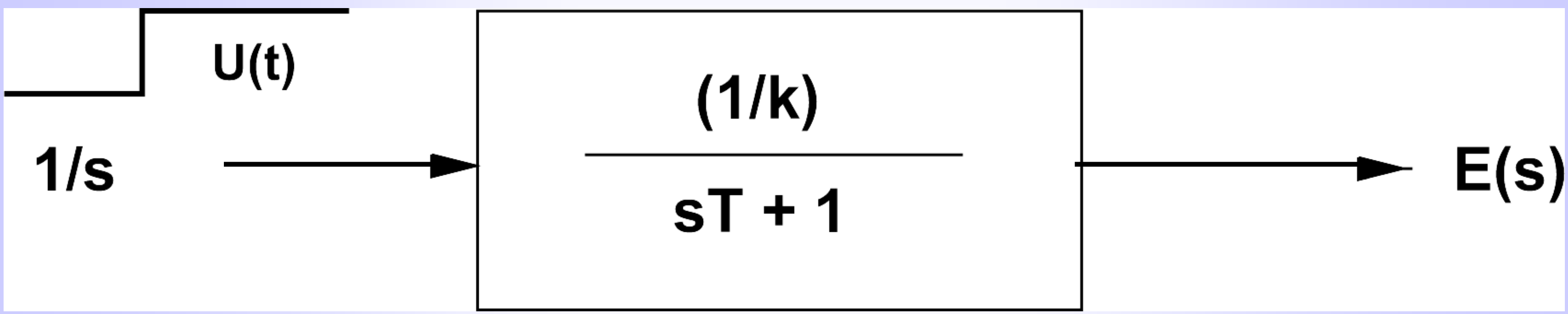
Laplace Transformation:

$$\Delta R_m(s) = k \cdot E(s) + r \cdot s \cdot E(s) = (k + r \cdot s) \cdot E(s)$$

The Laplace transfer function (output/input) of the plant is then:

$$H(s) \equiv \frac{E(s)}{\Delta R_m(s)} = \frac{1}{k + r \cdot s} = \frac{(1/k)}{1 + s \cdot (r/k)} = \frac{(1/k)}{1 + s \cdot T_E}$$

Example:
Compute the Step response for the 1st-order plant:



$$E(s) = \frac{1}{s} \cdot \frac{(1/k)}{1+sT} = \frac{(1/k)}{s} - \frac{(1/k)}{s+(1/T)}$$

$\xrightarrow{LT^{-1}}$

$$E(t) = \frac{1}{k} \cdot \left(1 - e^{-t/T}\right)$$

Important: first rewrite the Laplace function into its standard form before applying LT^{-1} !

Interestingly, we can get our results even faster!

For this, we only need to look at the poles and zeros of the LT:

$$H(s) \equiv \frac{\text{numerator}}{\text{denominator}} = \frac{N(s)}{D(s)}$$

Zeros: N(s) = 0
Poles: D(s) = 0

Because N(s) en D(s) are complex functions,
and H(s) is derived from an arbitrary order linear differential equation,
it follows that N(s) and D(s) can always be written as the product
of zero-, first-, and second-order polynomials in s:

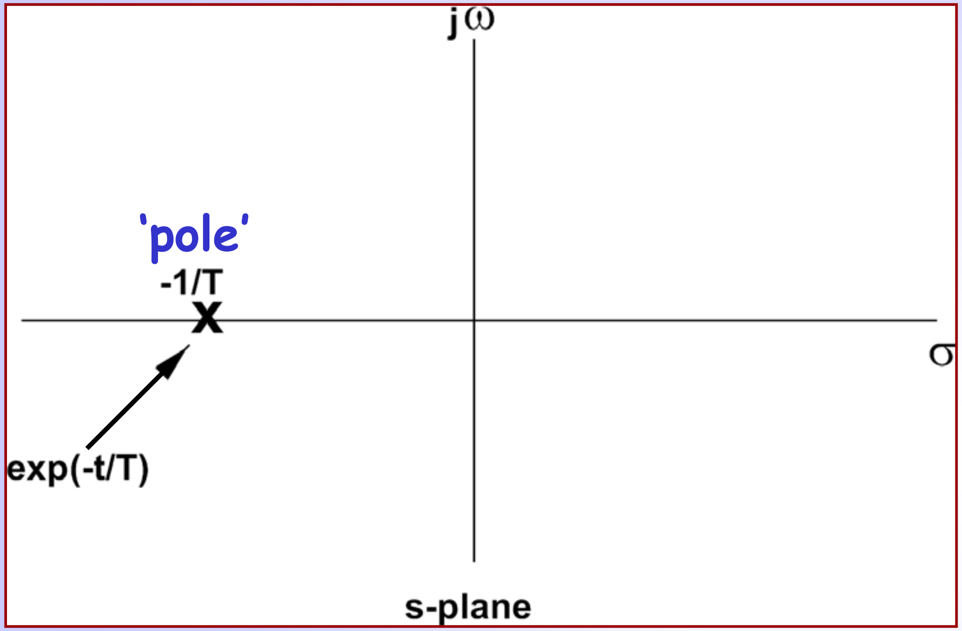
$$N(s) = \sum_{n=1}^{N} a_n \cdot s^n = (a + b \cdot s) \cdot (c + d \cdot s + e \cdot s^2) \cdots$$

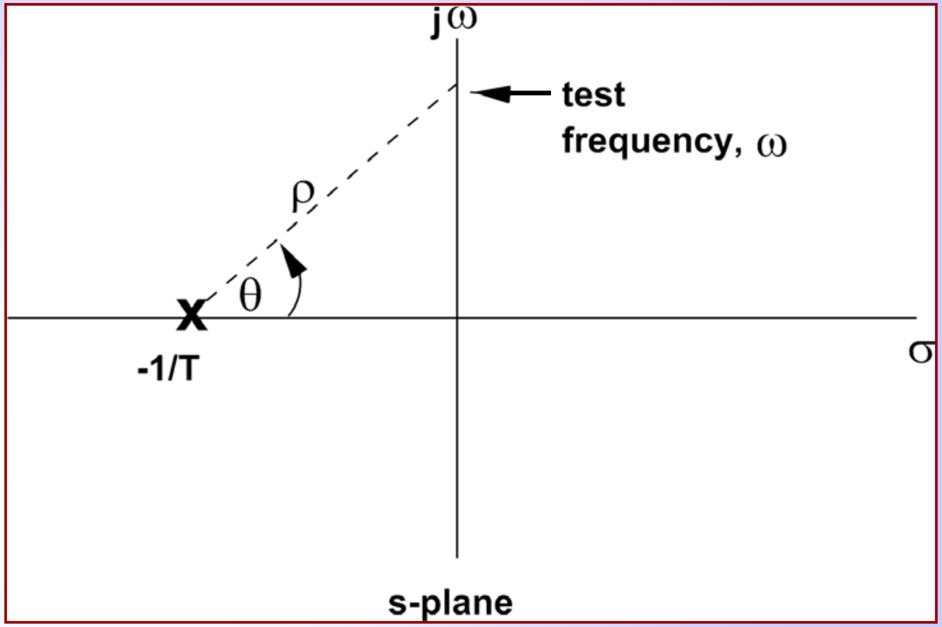
The 1st-order polynomials yield zeros or poles that lie on
real (sigma-)axis of s-space (i.e., exponential functions of time)
The 2nd-order polynomials always yield complex-conjugated pairs
as factors: $s_{1,2} = a \pm j \cdot \omega$

How does this work for our simple eye-plant model?

Impulse response of the plant:

$$H(s) \equiv \frac{E(s)}{\Delta R_m(s)} = \frac{(1/k)}{1 + s \cdot T_E}$$





Reading Gain- and Phase characteristics:

From the location of the pole, we can immediately write the impulse response function in the time domain, as well as the gain- and phase characteristics!

$$G(\omega) = \frac{1}{\rho}$$

$$\Phi(\omega) = -\theta$$

In general the following holds for more complex transfer functions:

1)Amplitude characteristics multiply with each other
2)Phase characteristics are added to each other (lag: -, advance: +).
3)Amplitude characteristic for a pole: 1/ρ(ω), and for a zero: ρ(ω)
4)Phase characteristic for a pole: negative angle θ(ω) (phase lag)
for a zero: positive angle θ(ω) (phase advance)

For example:

$$\boxed{H(s) \equiv \frac{1 + a \cdot s}{(1 + b \cdot s) \cdot (1 + c \cdot s)}}$$

than the zero is: s1 = -1/a
And two poles: s2=-1/b and s3=-1/c

In the frequency domain the transfer characteristic then becomes:

$$G(\omega) = \frac{\sqrt{\omega^2 + (1/a)^2}}{\sqrt{\omega^2 + (1/b)^2} \cdot \sqrt{\omega^2 + (1/c)^2}}$$ and

$$\Phi(\omega) = \arctan(a \cdot \omega) - \arctan(b \cdot \omega) - \arctan(c \cdot \omega)$$

Time behaviour of h(t) is found by inverse LT (after fraction splitting….): at WC!

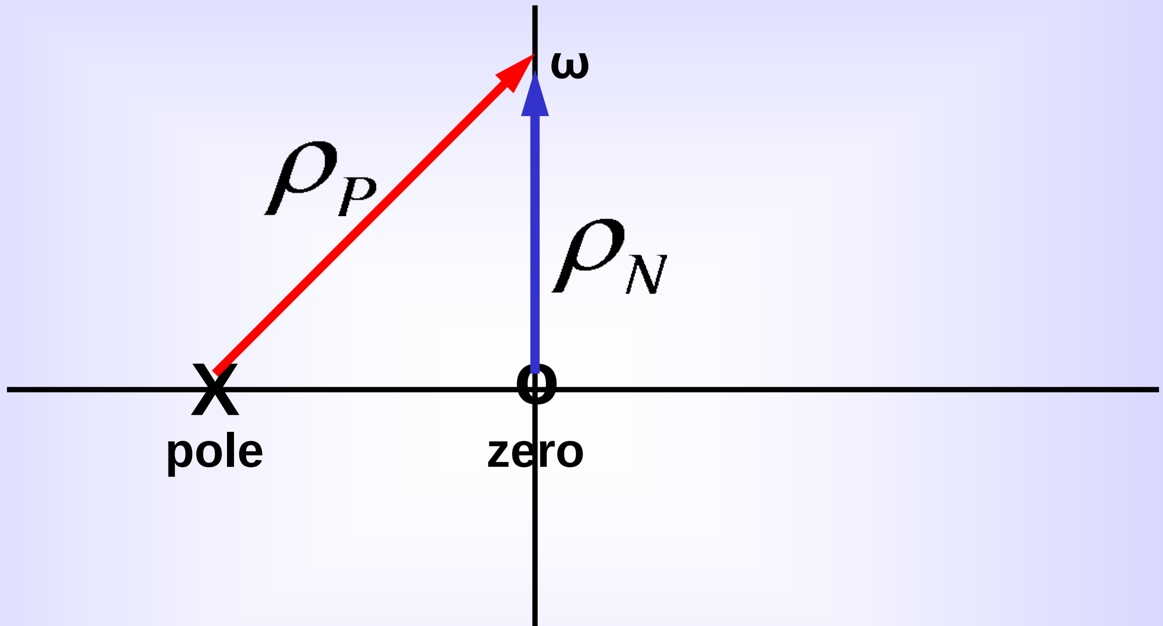
Model analysis through Laplace transformation:

Some examples:

- **the VOR**
- **'velocity storage'**
- **neural integration**

the VOR:

$$\frac{d\dot{H}}{dt} = \frac{1}{T_{vor}} \cdot \dot{E}(t) + \frac{d\dot{E}}{dt} \quad \xrightarrow{\text{LT}} \quad T(s) \equiv \frac{\dot{E}}{\dot{H}} = \frac{sT_{vor}}{1+sT_{vor}}$$



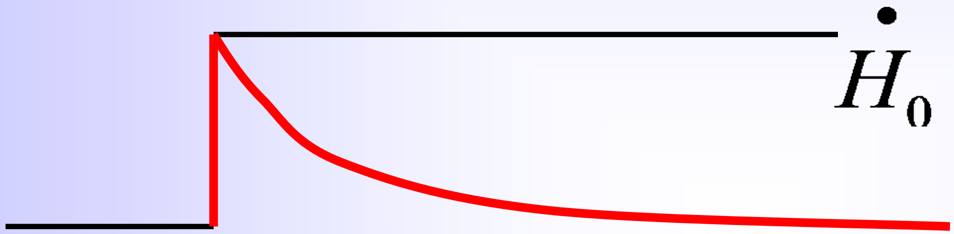
Transfer characteristic:

$$G_{vor}(\omega) = G_P(\omega) \bullet G_Z(\omega) = \frac{\rho_Z(\omega)}{\rho_P(\omega)} = \frac{\omega}{\sqrt{\omega^2 + (1/T)^2}}$$

$$\Phi(\omega) = \Phi_Z(\omega) - \Phi_P(\omega) = (\pi/2) - \arctan(\omega T)$$

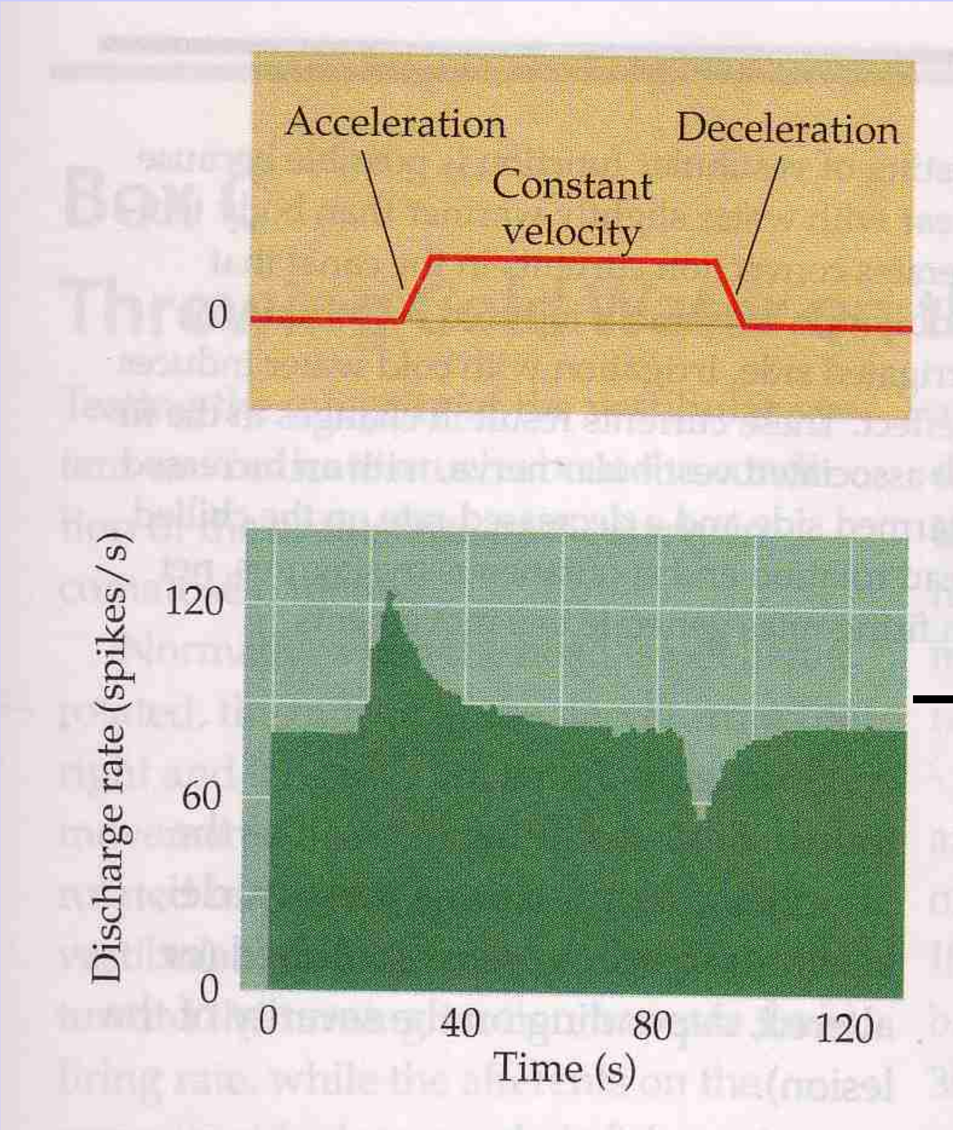
Step response of the VOR?

$$\dot{E}(s) = \frac{\dot{H}_0}{s} \times \frac{sT_{VOR}}{1 + sT_{VOR}} = \frac{\dot{H}_0 T_{VOR}}{1 + sT_{VOR}}$$



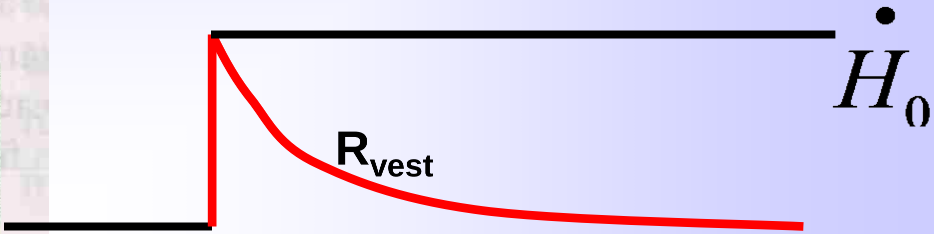
$$\boxed{\dot{E}(t) = \dot{H}_0 \cdot e^{-t/T_{VOR}}}$$

with $T_{VOR} \approx 21$ s



Recording of neural activity from the vestibular nerve (from Purves, Ch 13) during a constant-velocity head rotation

There appears to be an interesting problem in relation to the VOR:



$$R_{vest}(t) = \dot{H}_0 \cdot e^{-t/T_{can}}$$

with $T_{can} \approx 7$ s